SIAM J. MATH. ANAL.
Vol. 20, No. 1, January 1989

007

CAUCHY FORMULAE FOR FUNCTIONS ANALYTIC OF ORDER TWO ON C^1 DOMAINS WITH APPLICATIONS TO ELASTOSTATICS AND HYDROSTATICS*

JONATHAN COHEN†

Abstract. This paper gives Cauchy-type formulae for functions analytic of order two on C^1 domains obtained from the solutions of corresponding biharmonic problems. Functions analytic of order two are shown to be potentials for the solutions of systems of linear partial differential equations in two-dimensional elastostatics and hydrostatics. When combined with Cauchy formulae, integral representations are obtained for the traction problem and the linearized Stokes problem that are valid even for C^1 domains.

Key words. Cauchy formulae, analytic of order two, traction problem, stationary Stokes problem, biharmonic equation, layer potentials

AMS(MOS) subject classifications. 30, 35, 42, 45, 73, 76

1. Introduction. In this paper, we give Cauchy-type formulae for functions analytic of order two on C^1 domains obtained from the solutions of corresponding biharmonic problems. We show how functions analytic of order two are potentials for the solutions of systems of linear partial differential equations in two-dimensional elastostatics and hydrostatics. When combined with Cauchy formulae we obtain integral representations for the traction problem and the linearized Stokes problem that are valid even for C^1 domains.

The traction problem in elastostatics is to determine the components of stress and displacement from a system of partial differential equations satisfied within a domain and the forces applied on the boundary. Airy showed [2] that in the absence of body forces this problem could be reduced to finding a scalar biharmonic potential called the stress function. The stationary Stokes problem in hydrostatics is the following. Solve a system of partial differential equations satisfied by the velocity and pressure that can likewise be reduced to a biharmonic problem. (See Mikhlin [11, pp. 176–178] for an outline of these reductions.)

In [7] and [8] Cohen and Gosselin obtained solutions to the following biharmonic problems on C^1 domains in $\mathbb{R}^2$:

$$\begin{aligned} &\Delta^2 u = 0 \quad \text{in } \Omega, \\ &\nabla u|_{\partial\Omega} = \mathbf{g} \quad \text{where} \int \mathbf{g}\cdot\mathbf{T}\,ds = 0, \end{aligned} \tag{1.1}$$

$\mathbf{T}$ is the unit tangent vector and $\mathbf{g} \in L^p \times L^p(\partial\Omega)$, $1 < p < \infty$

$$\Delta^2 u = 0, \qquad (u_{xx}x_s + u_{xy}y_s,\ u_{xy}x_s + u_{yy}y_s) = \boldsymbol{\varphi} \tag{1.2}$$

where $\boldsymbol{\varphi} = (\varphi, \psi) \in L^q \times L^q(\partial\Omega)$ and $\int_{\partial\Omega} \varphi = \int_{\partial\Omega} \psi = \int_{\partial\Omega} x\varphi + y\psi\, ds = 0$.

The first of these problems involves Dirichlet-type boundary conditions and solves the stationary Stokes problem. The second involves adjoint or Neumann-type boundary conditions and solves the traction problem.

The solutions are given by potentials that are modified versions of the multiple layer potentials introduced by Agmon in [1]. The analysis at the boundary is obtained

---

* Received by the editors July 21, 1986; accepted for publication (in revised form) March 21, 1988.

† Department of Mathematics, De Paul University, Chicago, Illinois 60614.

for C^1 domains via an application of Calderon's theorem on the Cauchy integral along Lipschitz curves [5].

In practice it can be difficult to obtain the appropriate biharmonic potentials. Muskhelishvili's book on elasticity [12, Chap. 5] shows how assuming the existence of Airy's stress function leads to the reformulation of the traction problem as a boundary value problem for a system of analytic functions. The crucial point in the introduction of analytic functions is that biharmonic functions can be represented as $\operatorname{Re}\{\bar{z}f(z)+g(z)\}$ where f and g are analytic. If we define $\bar{\partial}=\partial_x+i\partial_y$, a simple calculation shows that $\bar{\partial}^2(\bar{z}f(z)+g(z))=0$. This suggests a connection between elasticity and the $\bar{\partial}^2$ equation.

In fact, this connection is neither new nor surprising. In the 1920s Burgatti, in [3] and [4], studied solutions of $\bar{\partial}^n f=0$ and introduced solutions of the equation $\bar{\partial}^2 f=0$ into the equations of elasticity to obtain Kolosoff's formula for the complex displacement [4, pp. 90–91].

In this paper we look at the complex function $U+i\tilde{U}$ where U is the Airy stress function and $\tilde{U}$ is a biharmonic conjugate of U in the sense that $\bar{\partial}^2(U+i\tilde{U})=0$. We show how the displacements can be computed, up to a rigid infinitesimal deformation, as a linear combination of the first derivatives of the stress function and its biharmonic conjugate $\tilde{U}$. We then show how the layer potential representation of the solution to the biharmonic reformulation of the traction problem can be extended to a "Cauchy type" formula that automatically produces the biharmonic conjugate of the stress function.

This procedure is simpler than the method outlined on pages 106–109 of Muskhelishvili [12]. Furthermore, for half planes with any orientation, the layer potential solutions reduce to Poisson integrals of the boundary forces.

2. The $\bar{\partial}^2$ equation. Functions satisfying the equation $\bar{\partial}^2\psi=0$ are called analytic of order two where $\bar{\partial}=\partial_x+i\partial_y$ and ψ is complex valued. In this section we review some of the basic properties of these functions, some of which are discussed in a more general context in the articles by Burgatti [3] and [4].

We let ∂ denote the operator $\partial_x-i\partial_y$ and assume that ψ is complex valued and satisfies $\bar{\partial}^2\psi(z)=0$ in a domain Ω. If we write $\psi=U+i\tilde{U}$ where U and $\tilde{U}$ are smooth real-valued functions and observe that $\bar{\partial}^2=\partial_{xx}-\partial_{yy}+2i\partial_{xy}$, then $\bar{\partial}^2\psi=0$ implies

$$U_{xx}-U_{yy}-2\tilde{U}_{xy}+i(2U_{xy}+\tilde{U}_{xx}-\tilde{U}_{yy})=0. \tag{2.1}$$

Equating real and imaginary parts, we obtain the following system of second-order partial differential equations:

$$U_{xx}-U_{yy}=2\tilde{U}_{xy}, \tag{2.2}$$

$$2U_{xy}=-(\tilde{U}_{xx}-\tilde{U}_{yy}), \tag{2.3}$$

which is analogous to the Cauchy–Riemann equations.

Since the Laplace operator can be factored as $\Delta=\bar{\partial}\partial$ we observe the following:

$$\Delta U+i\Delta\tilde{U} \quad \text{is analytic} \tag{2.4}$$

since $\bar{\partial}(\Delta U+i\Delta\tilde{U})=\partial\bar{\partial}^2(U+i\tilde{U})=0$;

$$\Delta^2 U=\operatorname{Re}\Delta^2(U+i\tilde{U})=\operatorname{Re}\partial^2\bar{\partial}^2(U+i\tilde{U})=0; \tag{2.5}$$

$$\Delta^2\tilde{U}=\operatorname{Im}\Delta^2(U+i\tilde{U})=\operatorname{Im}\partial^2\bar{\partial}^2(U+i\tilde{U})=0. \tag{2.6}$$

Thus U and $\tilde{U}$ are biharmonic and since $\Delta U + i\Delta\tilde{U}$ is analytic, the pair of functions ΔU and $\Delta\tilde{U}$ satisfy the Cauchy–Riemann equations:

$$(\Delta U)_x = (\Delta\tilde{U})_y, \tag{2.7}$$

$$(\Delta U)_y = -(\Delta\tilde{U})_x. \tag{2.8}$$

The system of real second-order equations (2.2) and (2.3) together with (2.8) and (2.9) will be referred to as the biharmonic Cauchy–Riemann equations.

Analytic functions of order two can be represented as

$$\psi(z) = \frac{\bar{z}}{2} f(z) + g(z) \tag{2.9}$$

where f and g are analytic. It follows that $f(z) = \bar{\partial}\psi(z)$ and $g(z) = \psi(z) - (\bar{z}/2)\bar{\partial}\psi(z)$. We then write

$$\psi(z) = \frac{\bar{z}}{2}\bar{\partial}\psi(z) + \left(\psi(z) - \frac{\bar{z}}{2}\bar{\partial}\psi(z)\right), \tag{2.10}$$

and applying Cauchy's formula to the analytic functions $\bar{\partial}\psi(z)$ and $\psi(z) - (\bar{z}/2)\bar{\partial}\psi(z)$, we obtain the Cauchy-type representation

$$\psi(z) = \frac{1}{2\pi i}\int_\gamma \frac{\psi(w)}{w-z}\,dw - \frac{1}{4\pi i}\int_\gamma \bar{\partial}\psi(w)\,\frac{\bar{w}-\bar{z}}{w-z}\,dw \tag{2.11}$$

where γ is a simple closed contour in Ω containing z.

Formula (2.11) appears (except for a small error) on page 88 of Burgatti [4], and we will refer to (2.11) as Burgatti's formula.

3. The traction problem. The traction problem is to obtain the elastostatic state of a thin plate from the forces at the edge. In this section we assume that Ω is a bounded C^1 simply connected domain in $\mathbb{R}^2$. We let $S = \left[\begin{smallmatrix} A & B \\ B & C \end{smallmatrix}\right]$ denote the stress tensor and let u, v denote the components of displacement. In the absence of body forces the equilibrium equations are

$$A_x + B_y = 0, \qquad B_x + C_y = 0 \tag{3.1}$$

and the equations relating displacement and stress are

$$A = (\lambda + 2\mu)u_x + \lambda v_y, \qquad B = \mu(u_y + v_x), \qquad C = \lambda u_x + (\lambda + 2\mu)v_y. \tag{3.2}$$

If we let (X_n, Y_n) denote the normal stress along the boundary $\partial\Omega$ and (x_s, y_s) the unit tangent vector on $\partial\Omega$, then the traction problem is to find components of stress A, B, C and displacements u, v satisfying (3.1) and (3.2) in Ω and for which

$$\begin{bmatrix} A & B \\ B & C \end{bmatrix}\begin{bmatrix} y_s \\ -x_s \end{bmatrix} = \begin{bmatrix} X_n \\ Y_n \end{bmatrix} \quad \text{on } \partial\Omega.$$

Airy showed that the equilibrium equations (3.1) imply the existence of a function w such that $A = w_{yy}$, $B = -w_{xy}$, and $C = w_{xx}$. Furthermore, substituting the second partials of w for A, B, and C into (3.2), it easily follows that $\Delta^2 w = 0$. This means that the traction problem can be reformulated as the biharmonic problem:

$$\begin{aligned} &\Delta^2 w = 0, \\ &\begin{bmatrix} w_{yy} & -w_{xy} \\ -w_{xy} & w_{xy} \end{bmatrix}\begin{bmatrix} y_s \\ -x_s \end{bmatrix} = \begin{bmatrix} X_n \\ Y_n \end{bmatrix} \quad \text{in } \partial\Omega. \end{aligned} \tag{3.3}$$

A quick inspection shows that this is problem (1.2) in the Introduction. The solution w is called the Airy stress function.

To complete the solution to the traction problem it is necessary to compute the displacements from the stress function. This procedure, as outlined in Muskhelishvili [12, pp. 106–109], is somewhat complicated. We will now show how, if a biharmonic conjugate can be found for the stress function, the computation can be simplified considerably.

If we substitute the appropriate second partials of w for A and C in the first and third equations of (3.1) and solve for u_x and v_y we get

$$u_x = \frac{-\lambda w_{xx} + (\lambda + 2\mu) w_{yy}}{4\mu(\lambda + \mu)}, \tag{3.4}$$

$$v_y = \frac{(\lambda + 2\mu) w_{xx} - \lambda w_{yy}}{4\mu(\lambda + \mu)}. \tag{3.5}$$

We now assume there exists a function $\tilde{w}$ such that $\bar{\partial}^2(w + i\tilde{w}) = 0$. Using the biharmonic Cauchy-Riemann equations we can substitute $w_{xx} - 2\tilde{w}_{xy}$ for w_{yy} in (3.4) and $w_{yy} + 2\tilde{w}_{xy}$ for w_{xx} in (3.5) to get

$$u_x = \frac{\partial}{\partial x}\left(\frac{\mu w_x - (\lambda + 2\mu)\tilde{w}_y}{2\mu(\lambda + \mu)}\right), \tag{3.6}$$

$$v_y = \frac{\partial}{\partial x}\left(\frac{\mu w_y + (\lambda + 2\mu)\tilde{w}_x}{2\mu(\lambda + \mu)}\right). \tag{3.7}$$

This is the main point. The introduction of the biharmonic conjugate $\tilde{w}$ enables us to integrate u_x and v_y to obtain

$$u = \frac{\mu w_x - (\lambda + 2\mu)\tilde{w}_y}{2\mu(\lambda + \mu)} + F_1(y), \tag{3.8}$$

$$v = \frac{\mu w_y + (\lambda + 2\mu)\tilde{w}_x}{2\mu(\lambda + \mu)} + F_2(x). \tag{3.9}$$

Substituting for u and v in (3.2), we have

$$\begin{aligned} \frac{-1}{\mu} w_{xy} &= (u_y + v_x) \\ &= \frac{\mu w_{xy} - (\lambda + 2\mu)\tilde{w}_{yy}}{2\mu(\lambda + \mu)} + F_1'(y) + \frac{\mu w_{xy} - (\lambda + 2\mu)\tilde{w}_{xx}}{2\mu(\lambda + \mu)} + F_2'(x) \\ &= \frac{2\mu w_{xy} - (\lambda + 2\mu)(\tilde{w}_{yy} - \tilde{w}_{xx})}{2\mu(\lambda + \mu)} + F_1'(y) + F_2'(x), \end{aligned} \tag{3.10}$$

which, by the biharmonic Cauchy-Riemann equations,

$$\begin{aligned} &= \frac{2\mu w_{xy} - (\lambda + 2\mu)(2w_{xy})}{2\mu(\lambda + \mu)} + F_1'(y) + F_2'(x) \\ &= -\frac{1}{\mu} w_{xy} + F_1'(y) + F_2'(x). \end{aligned}$$

Thus we have $0 = F_1'(y) + F_2'(x)$. This implies that $F_1'(y) = -F_2'(x) = \varepsilon$ and so $F_1(y) = \varepsilon y + \tau$, $F_2(x) = -\varepsilon x + \sigma$.

The choice of biharmonic conjugate $\tilde{w}$ was arbitrary. However, it follows from Lemma 13.1 of Agmon [1] that if $\bar{\partial}^2(w+i\tilde{w})=0$ and $\bar{\partial}^2(w+i\tilde{w}_1)=0$, then $\tilde{w}_1-\tilde{w}=\alpha x+\beta y+\frac{1}{2}(x^2+y^2)+\delta$. Substituting $\tilde{w}_1$ for $\tilde{w}$ in (3.8) and (3.9), we get a displacement (u_1, v_1) that differs from (u, v) by

$$u_1-u=\frac{-(\lambda+2\mu)}{2\mu(\lambda+\mu)}(\gamma y+\beta), \tag{3.11}$$

$$v_1-v=\frac{(\lambda+2\mu)}{2\mu(\lambda+\mu)}(\gamma x+\alpha). \tag{3.12}$$

This means that two distinct "biharmonic conjugates" give rise to displacements that differ by, at worst, an infinitesimal rigid displacement. In other words, we have introduced no new pure deformation by computing the displacement from the derivatives of $\tilde{w}_1$ rather than $\tilde{w}$. Up to an infinitesimal rigid displacement we have the formula

$$(u, v)=\left(\frac{\mu w_x-(\lambda+2\mu)\tilde{w}_y}{2\mu(\lambda+\mu)}, \frac{\mu w_y+(\lambda+2\mu)\tilde{w}_x}{2\mu(\lambda+\mu)}\right). \tag{3.13}$$

4. The stationary Stokes problem. The stationary Stokes problem in hydrostatics has the formulation in a domain Ω:

$$\begin{aligned} &\Delta \mathbf{u}=\nabla p \quad \text{in } \Omega \\ &\operatorname{div} \mathbf{u}=0 \quad \text{in } \Omega, \\ &\mathbf{u}|_{\partial\Omega}=\mathbf{f} \end{aligned} \tag{4.1}$$

where $\mathbf{u}=(u, v)$ is the velocity of the fluid, p is the pressure, and $\mathbf{f}=(f_1, f_2)$ is the velocity at the boundary. The second equation, $\operatorname{div} \mathbf{u}=0$, implies there exists a function Φ satisfying $\nabla\Phi=(-v, u)$. It then follows from substitution for u and v in the first equation that $\Delta^2\Phi=\operatorname{div}(\Delta\Phi_x, \Delta\Phi_y)=-p_{xy}+p_{xy}=0$. The Stokes problem then has the biharmonic formulation:

$$\begin{aligned} &\Delta^2\Phi=0 \quad \text{in } \Omega, \\ &\nabla\Phi|_{\partial\Omega}=(-f_2, f_1). \end{aligned} \tag{4.2}$$

It remains to obtain the pressure p from the solution Φ of (4.2). If we assume there exists a $\tilde{\Phi}$ such that $\bar{\partial}^2(\Phi+i\tilde{\Phi})=0$, then by the second part of the biharmonic Cauchy-Riemann equations,

$$(\Delta\tilde{\Phi})_x=-(\Delta\Phi)_y=-\Delta u=-p_x, \qquad (\Delta\tilde{\Phi})_y=(\Delta\Phi)_x=-\Delta v=-p_y. \tag{4.3}$$

Hence $\nabla(-\Delta\tilde{\Phi})=\nabla p$ so that $-\Delta\tilde{\Phi}$ differs from the pressure by a constant. If a second biharmonic conjugate $\tilde{\Phi}_1$ is used, then $\Delta\tilde{\Phi}_1-\Delta\tilde{\Phi}$ is a constant.

It is clear that this same calculation shows that any harmonic conjugate of $\Delta\Phi$ will suffice to give the pressure. However, we will point out that the layer potential solution of (4.2) automatically produces a biharmonic conjugate $\tilde{\Phi}$ so that no additional integrations are necessary to find the pressure.

5. The biharmonic results. We assume that Ω is a bounded, simply connected C^1 domain in R^2 with boundary $\partial\Omega$. We next introduce the following spaces of boundary data.

DEFINITION 5.1.

$$C_p=\left\{\mathbf{g}=(g, h)\in(L^p\times L^p)(\partial\Omega): \int_{\partial\Omega} g\,dx+h\,dy=0\right\}.$$

DEFINITION 5.2. $(L^p \times L^p)_0(\partial\Omega) = \{\boldsymbol{\varphi} = (\varphi, \psi) \in (L^q \times L^q)(\partial\Omega) : \int_{\partial\Omega} \nabla w \cdot \boldsymbol{\varphi}\, ds = 0$ for all $w(x, y) = \alpha x + \beta y + \gamma(x^2 + y^2) + \delta$ and $1 < q < \infty\}$.

Let $\tilde{F}(x, y) = (-1/4\pi)\{(x^2 + y^2) \arg(x + iy) - xy\}$ for some particular choice of the argument. In what follows we will let X denote points in the domain Ω and P, and Q will denote points on the boundary.

DEFINITION 5.3. For $Q \in \partial\Omega$ and $X \in \Omega$ we define the boundary differential operator $\mathbf{L} = \mathbf{L}_Q$ by

$$\mathbf{L}v(X) = (L_1 v(X), L_2 v(X))$$

where

$$L_1 v(X) = v_{xx}(X)x_s(Q) + v_{xy}(X)y_s(Q), \qquad L_2 v(X) = v_{xy}(X)x_s(Q) + v_{yy}(X)y_s(Q) \tag{5.4}$$

with $(x(s), y(s))$ being the arclength parameterization of $\partial\Omega$ and $(x_s(Q), y_s(Q))$ being the unit tangent at Q.

DEFINITION 5.5. For $\mathbf{g} \in C_p$ we define the modified multiple layer potential by

$$u_m(\mathbf{g} : X) = \int_{\partial\Omega} g(Q) L_1 \tilde{F}(X - Q) + h(Q) L_2 \tilde{F}(X - Q)\, ds(Q). \tag{5.6}$$

For $\boldsymbol{\varphi} \in (L^q \times L^q)_0(\partial\Omega)$ we define the modified lower-order potential by

$$v_m(\boldsymbol{\varphi}; X) = \int_{\partial\Omega} \varphi(P) \tilde{F}_x(P - X) + \psi(P) \tilde{F}_y(P - X)\, ds(P). \tag{5.7}$$

For $X \notin \partial\Omega$ we can differentiate (5.6) and (5.7) under the integral signs to get

$$\nabla u_m = \int_{\partial\Omega} \mathbf{g}(Q) l(X, Q)\, ds(Q) \tag{5.8}$$

where

$$l(X, Q) = \begin{bmatrix} \partial_x^X L_1 \tilde{F}(X - Q) & \partial_y^X L_1 \tilde{F}(X - Q) \\ \partial_x^X L_2 \tilde{F}(X - Q) & \partial_y^X L_2 \tilde{F}(X - q) \end{bmatrix} \tag{5.9}$$

and

$$\mathbf{L} v_m(X)^T = \int_{\partial\Omega} l(X, P, Q) \boldsymbol{\varphi}(P)^T\, ds(P) \tag{5.10}$$

where

$$l(X, P, Q) = \begin{bmatrix} \partial_x^P L_1 \tilde{F}(P - X) & \partial_y^P L_1 \tilde{F}(P - X) \\ \partial_x^P L_2 \tilde{F}(P - X) & \partial_y^P L_2 \tilde{F}(P - X) \end{bmatrix}, \tag{5.11}$$

and the superscript T denotes the transpose of a row vector. Note that the dependence of l on Q is built into the definition of L_1 and L_2.

DEFINITION 5.12. For $P \neq Q$, $P, Q \in \partial\Omega$, we can define the matrix kernels in (5.9) and (5.11) by letting $X = Q$. Both kernels are then the same and we call them $l(P, Q)$. We define the operators

$$\mathscr{L}_\varepsilon \mathbf{g}(P) = \int_{|P-Q|>\varepsilon} \mathbf{g}(Q, Q) l(P, Q)\, ds(Q) \tag{5.13}$$

and

$$\mathscr{L}_\varepsilon^* \boldsymbol{\varphi}(P) = \int_{|P-Q|>\varepsilon} l(P, Q) \boldsymbol{\varphi}(P)^T\, ds(P). \tag{5.14}$$

We tentatively define the operators $\mathscr{L}\mathbf{g}(P)=\lim_{\varepsilon\to 0}\mathscr{L}_\varepsilon\mathbf{g}(P)$ and $\mathscr{L}^*\varphi(Q)=\lim_{\varepsilon\to 0}\mathscr{L}_\varepsilon^*\varphi(Q)$.

THEOREM 5.15. *For* $\mathbf{g}\in C_p$, *we have the following*:

(i) $\mathscr{L}\mathbf{g}(P)$ *exists almost everywhere with respect to arclength,* $\mathscr{L}$ *is bounded from* C_p *to itself in the* $(L^p\times L^p)(\partial\Omega)$ *norm and in fact is compact from* C_p *to itself.*

(ii) *The nontangential*

$$\lim_{X\to P}\nabla u_m(X)=\begin{cases}(I+\mathscr{L})\mathbf{g}(P), & X\in\Omega,\\ (-I+\mathscr{L})\mathbf{g}(P), & X\notin\bar{\Omega}\end{cases}$$

for almost every $P\in\partial\Omega$,

(iii) $(I+\mathscr{L})^{-1}$ *exists on* C_p *and* $(-I+\mathscr{L})^{-1}$ *exists on the space* $(-I+\mathscr{L})(C_p)$.

COROLLARY 5.16. *The interior Dirichlet problem* $\Delta^2u=0$ *in* Ω, $\nabla u=\mathbf{g}\in C_p$ *on* $\partial\Omega$ *is solvable with* $u=u_m((I+\mathscr{L})^{-1}\mathbf{g};X)$. The exterior Dirichlet problem $\Delta^2u=0$ *in* $\bar{\Omega}^C$, $\nabla u=\mathbf{g}\in C_p$ *on* $\partial\Omega$ *is solvable with* $u=u_m((-I+\mathscr{L})^{-1}\mathbf{g}_0;X)+\nabla w$ *where for* $\mathbf{g}\in C_p$, $\mathbf{g}$ *can be written uniquely as* $\mathbf{g}=\mathbf{g}_0+\nabla w$ *with* $\mathbf{g}_0\in(-I+\mathscr{L})(C_p)$ *and* $w=\alpha x+\beta y+\gamma(x^2+y^2)+\delta$.

THEOREM 5.17. *For* $\varphi\in(L^q\times L^q)_0(\partial\Omega)$ *we have the following*:

(i) $\mathscr{L}^*\varphi(Q)$ *exists almost everywhere with respect to arclength,* $\mathscr{L}$ *is bounded from* $(L^q\times L^q)_0(\partial\Omega)$ *to itself in the* $(L^q\times L^q)(\partial\Omega)$ *norm and is compact from* $(L^q\times L^q)_0(\partial\Omega)$ *to itself.*

(ii) *The nontangential*

$$\lim_{X\to Q}\mathbf{L}v_m(X)=\begin{cases}(-I+\mathscr{L}^*)\varphi(Q), & X\in\Omega,\\ (I+\mathscr{L}^*)\varphi(Q), & X\notin\bar{\Omega}\end{cases}$$

for almost every $Q\in\partial\Omega$.

(iii) $(-I+\mathscr{L}^*)^{-1}$ *exists on* $(L^q\times L^q)_0(\partial\Omega)$.

COROLLARY 5.18. *The adjoint boundary value problem* $\Delta^2v=0$ *in* Ω, $\mathbf{L}v=\varphi\in(L^q\times L^q)_0(\partial\Omega)$ *is solvable by* $v=v_m((-I+\mathscr{L}^*)^{-1}\varphi;X)$.

Remark. It is important to note that the operator $-I+\mathscr{L}^*$ is not exactly the adjoint of $-I+\mathscr{L}$. The adjoint of $-I+\mathscr{L}$ acts on the dual of C_p, which is a coset space. The space $(L^q\times L^q)_0(\partial\Omega)$ is a function space that is close to the dual of C_p, however, work is required to extend the invertibility of the adjoint of $-I+\mathscr{L}$ from the dual of C_p to invertibility on $(L^q\times L^q)_0(\partial\Omega)$. The details of the proofs can be found in Cohen and Gosselin [7] and [8].

6. Cauchy formulae. In the theory of Hardy spaces of the upper half plane, functions $f\in L^p(\mathbb{R})$, $1<p<\infty$ can be identified with analytic functions $f(z)$ on the upper half plane satisfying $\sup_{y>0}\int|f(x+iy)|^p_{dx}<\infty$. This identification is obtained by convolving the boundary function f with the complex kernel $(i\pi z)^{-1}$. For an arbitrary C^1 (or even a Lipschitz domain if $f\in L^p(\partial\Omega)$ for $p\geqq 2$) we can obtain the same type of identification by applying the properties of the classical double layer potential to the Cauchy integral of the boundary data. (See Fabes, Jodeit, and Riviere [9], Fabes and Kenig [10], or Verchota [13] for more details.)

An analogous kind of identification of compatible triples of boundary functions with analytic functions of order two can be obtained from Burgatti's formula (2.11). (By compatible triples we mean the space $\mathscr{B}_p=\{\dot{f}=(f,g,h)\in L_1^p\times L^p\times L^p(\partial\Omega): f_s=gx_s+hy_s$ almost everywhere with respect to arclength$\}$.) For $\dot{f}=(f,g,h)\in\mathscr{B}_p$ we define the complex potential

$$\varphi_{\dot{f}}(w)=\frac{1}{2\pi i}\int_{\partial\Omega}\frac{f(z)}{z-w}\,dz-\frac{1}{4\pi i}\int_{\partial\Omega}(g(z)+ih(z))\frac{\bar{z}-\bar{w}}{z-w}\,dz. \tag{6.1}$$

It follows from the article by Cohen and Gosselin on the Dirichlet problem for the biharmonic equation [6] that if $\mathcal{U}_{\dot{f}} = \operatorname{Re} \varphi_{\dot{f}}$ and $\dot{\mathcal{U}}_{\dot{f}} = (\mathcal{U}_{\dot{f}}, (\mathcal{U}_p)_x, (\mathcal{U}_p)_y)$, then there exists an invertible operator $T: \mathcal{B}_p \to \mathcal{B}_p$ such that the nontangential $\lim_{X \to P \in \partial\Omega} \dot{\mathcal{U}}_{T^{-1}\dot{f}}(X) = \dot{f}(P)$ almost everywhere. The map $\dot{f} \to \varphi_{\dot{f}}$ then identifies the boundary space $\mathcal{B}_p$ with a space of functions analytic of order two in Ω.

We have seen in §§ 3 and 4 that the solutions to the traction problem and stationary Stokes problem can be obtained by solving a biharmonic problem and finding a biharmonic conjugate. An examination of the matrix kernels (5.8) and (5.10) suggests that we can obtain biharmonic conjugates to the solutions of (1.1) and (1.2) from a Cauchy-type formula if we can find a biharmonic conjugate to the function $\tilde{F}$.

But $\tilde{F}(z) = (-1/4\pi) \operatorname{Im} \{\bar{z}z \log z - \frac{1}{2}\bar{z}z + \frac{1}{2}z^2\}$ and $\bar{\partial}^2\{\bar{z}z \log z - \frac{1}{2}\bar{z}z + \frac{1}{2}z\} = 0$. If we let $F(z) = \operatorname{Re} (-1/4\pi)\{\bar{z}z \log z - \frac{1}{2}\bar{z}z + \frac{1}{2}z^2\}$, then $\bar{\partial}^2(F + i\tilde{F}) = 0 = i\bar{\partial}^2(\tilde{F} - iF)$. This implies that $\tilde{F} - iF$ is analytic of order two, which suggests the following Cauchy-type potentials.

Definition 6.2. For $\mathbf{g} \in C_p$ define the complex modified multiple layer potential

$$(u_m + i\tilde{u}_m)(\mathbf{g}; X) = \int_{\partial\Omega} \mathbf{g}(Q)\mathbf{L}^Q(\tilde{F} - iF)(X - Q)^T \, ds(Q). \tag{6.3}$$

Definition 6.4. For $\varphi \in (L^q \times L^q)_0(\partial\Omega)$ define the complex modified lower-order potential

$$(v_m + i\tilde{v}_m)(\varphi, X) = \int_{\partial\Omega} \varphi(\beta)(\tilde{F} - iF)_x(P - X) + \psi(P)(\tilde{F} - iF)(P - X) \, ds(P). \tag{6.5}$$

It then follows immediately that $(u_m + i\tilde{u}_m)((I + \mathcal{L})^{-1}(f_2, -f_1); X)$ gives the function analytic of order two, which solves the stationary Stokes problem and $(v_m + i\tilde{v}_m) \cdot ((-I + \mathcal{L}^*)^{-1}(-Y_n, X_n); X)$ solves the traction problem. That is, all dependent variables can be obtained from the real and imaginary parts of these complex potentials by differentiation. Furthermore, explicit integral representations of the stresses, displacements, velocity components, and pressure can be obtained from taking the appropriate derivatives of the matrix kernels.

REFERENCES

[1] S. Agmon, *Multiple layer potentials and the Dirichlet problem for higher order elliptic equations in the plane*, Commun. Pure Appl. Math., 10 (1957), pp. 179–230.

[2] Airy, Brit., Assoc. Rep., 1862.

[3] P. Burgatti, *Sullè funzioni analitiche d'ordine n*, Boll. Un. Mat. Ital., 1922.

[4] ———, *Sulle funzioni analitiche d'ordine n e sull' equilibria elastico in due dimensioni*, Boll. Un. Mat. Ital., 1923.

[5] A. P. Calderon, *Cauchy integrals on Lipschitz curves and related operators*, Proc. Nat. Acad. Sci. U.S.A., 75, (1977), pp. 1324–1327.

[6] J. Cohen and J. Gosselin, *The Dirichlet problem for the biharmonic equation in a bounded C^1 domain in the plane*, Indiana Math. J., 32 (1983), pp. 635–685.

[7] ———, *Adjoint boundary value problems for the biharmonic equations on C^1 domains in the plane*, Ark. Mat., 23 (1985), pp. 217–240.

[8] ———, *Stress potentials on C^1 domains*, J. Math. Anal. Appl., to appear.

[9] E. Fabes, M. Jodeit, and N. Riviere, *Potential techniques for boundary value problems on C^1 domains*, Acta Math., (1979).

[10] E. Fabes and C. Kenig, *On the Hardy space of a C^1 domain*, Ark. Mat., 19 (1981), pp. 1–22.

[11] S. G. Mikhlin, *Integral Equations*, Pergamon Press, 1957.

[12] N. Muskhelishvili, *Some basic problems of the mathematical theory of elasticity*, P. Noordhoff, Groningen, the Netherlands, 1963.

[13] G. Verchota, *Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains*, J. Funct. Anal., 59 (1984), pp. 572–611.